

# SHATROVSKIĬ'S CONSTRUCTION OF THIN BASES

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ABSTRACT. The set  $\mathcal{A}$  of nonnegative integers is called a basis of order  $h$  if every nonnegative integer can be represented as the sum of exactly  $h$  not necessarily distinct elements of  $\mathcal{A}$ . An additive basis  $\mathcal{A}$  of order  $h$  is called thin if there exists  $c > 0$  such that the number of elements of  $\mathcal{A}$  not exceeding  $x$  is less than  $cx^{1/h}$  for all  $x \geq 1$ . This paper describes a construction of Shatrovskiĭ of thin bases of order  $h$ .

## 1. ADDITIVE BASES OF FINITE ORDER

Let  $\mathbf{N}_0$  denote the set of nonnegative integers, let  $h \geq 2$ , and let  $(\mathcal{A}_1, \dots, \mathcal{A}_h)$  be an  $h$ -tuple of subsets of  $\mathbf{N}_0$ . We define the *sumset*

$$\mathcal{A}_1 + \dots + \mathcal{A}_h = \{a_1 + \dots + a_h : a_i \in \mathcal{A}_i \text{ for } i = 1, \dots, h\}.$$

The  $h$ -fold *sumset* of a set  $\mathcal{A}$  of nonnegative integers is

$$h\mathcal{A} = \underbrace{\mathcal{A} + \dots + \mathcal{A}}_{h \text{ terms}}.$$

The  $h$ -tuple  $(\mathcal{A}_1, \dots, \mathcal{A}_h)$  is called an *additive system* of order  $h$  if  $\mathcal{A}_1 + \dots + \mathcal{A}_h = \mathbf{N}_0$ . The set  $\mathcal{A}$  is an *additive basis*, or, simply, a *basis*, of order  $h$  if  $h\mathcal{A} = \mathbf{N}_0$ . If  $(\mathcal{A}_1, \dots, \mathcal{A}_h)$  is an additive system of order  $h$ , then  $\mathcal{A} = \bigcup_{i=1}^h \mathcal{A}_i$  is a basis of order  $h$ .

For example, let  $a'_1, \dots, a'_h$  be relatively prime positive integers, and let

$$\mathcal{A}_i = a'_i * \mathbf{N}_0 = \{a'_i v_i : v_i \in \mathbf{N}_0\}$$

for  $i = 1, \dots, h$ . There exists an integer  $C$  such that every integer  $n \geq C$  can be represented in the form  $n = a'_1 v_1 + \dots + a'_h v_h$  with  $v_1, \dots, v_h \in \mathbf{N}_0$  (Nathanson [8, Section 1.6]), and so the set

$$\mathcal{A} = [0, C-1] \cup \bigcup_{i=1}^h \mathcal{A}_i$$

is an additive basis of order  $h$ .

Let  $g \geq 2$ . We can use the  $g$ -adic representation of the nonnegative integers to construct another class of additive bases of order  $h$ . Let  $K_1, \dots, K_h$  be subsets of  $\mathbf{N}_0$  such that

$$\mathbf{N}_0 = K_1 \cup \dots \cup K_h.$$

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For  $i = 1, \dots, h$ , let  $\mathcal{F}(K_i)$  denote the set of all finite subsets of  $K_i$ , and let

$$\mathcal{A}_i = \left\{ \sum_{k \in F} \varepsilon_k g^k : \varepsilon_k \in \{0, 1, 2, \dots, g-1\} \text{ and } F \in \mathcal{F}(K_i) \right\}.$$

Then  $(\mathcal{A}_1, \dots, \mathcal{A}_h)$  is an additive system of order  $h$  and  $\mathcal{A} = \bigcup_{i=1}^h \mathcal{A}_i$  is a basis of order  $h$ .

The *counting function* of a set  $\mathcal{A}$  of integers is

$$A(x) = \sum_{\substack{a \in \mathcal{A} \\ 1 \leq a \leq x}} 1.$$

If  $\mathcal{A}$  is a basis of order  $h$ , then for all  $x \in \mathbf{N}_0$  we have

$$x+1 \leq \binom{A(x)+h}{h} \leq \frac{(A(x)+h)^h}{h!}$$

and so

$$\liminf_{x \rightarrow \infty} \frac{A(x)}{x^{1/h}} \geq (h!)^{1/h}.$$

The basis  $\mathcal{A}$  of order  $h$  is called a *thin basis* if

$$\limsup_{x \rightarrow \infty} \frac{A(x)}{x^{1/h}} < \infty.$$

One of the first problems in additive number theory for general sets of integers concerned the existence and construction of thin bases of finite order (Rohrbach [11]). In 1937, Raikov [10] and Stöhr [14] independently solved this problem. Using the 2-adic representation and the sets  $K_i = \{k \in \mathbf{N}_0 : k \equiv i-1 \pmod{h}\}$  for  $i = 1, \dots, h$ , they proved that  $\mathcal{A} = \bigcup_{i=1}^h \mathcal{A}_i$  is a thin basis of order  $h$ . Many thin bases have been constructed using variations of the  $g$ -adic construction, for example, Cassels [2, 4, 9] and Jia-Nathanson [6]. Recently, Hofmeister [5], Blomer [1], and Schmitt [12] constructed other examples of thin bases.

In 1940 Shatrovskii [13] described a beautiful class of thin bases of order  $h$  that use the linear diophantine equation  $n = a'_1 x_1 + \dots + a'_h x_h$  and do not depend on  $g$ -adic representations. This is an almost forgotten paper.<sup>1</sup> It has been mentioned in a few surveys of solved and unsolved problems in additive number theory (e.g. Stöhr [15], Erdős and Nathanson [3], and Nathanson [7]), but MathSciNet lists no publication that cites Shatrovskii's paper. This note provides an exposition and generalization of Shatrovskii's construction of thin bases.

## 2. THE CONSTRUCTION

Fix an integer  $h \geq 2$ , and define the positive integer

$$(1) \quad k_0 = \left\lfloor \frac{1}{2^{1/h} - 1} \right\rfloor + 1.$$

Then  $1 + (1/k) < 2^{1/h}$  for all  $k \geq k_0$ . Let  $r_1, \dots, r_h$  be pairwise relatively prime positive integers, and arrange them in increasing order so that  $r_i < r_{i+1}$  for  $i = 1, \dots, h-1$ . Let  $P$  be a positive integer such that (i)  $P \geq r_h - r_1$ , and (ii) if

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<sup>1</sup>Shatrovskii published his paper in Russian, with a summary in French. The transliteration of the author's name in the French summary is Chatrovsky, and this is the form of the name that appears in *Mathematical Reviews*. I use the AMS and Library of Congress system for transliteration of Cyrillic into English.

$1 \leq i < j \leq h$  and if the prime  $p$  divides  $r_j - r_i$ , then  $p$  divides  $P$ . For every positive integer  $k$ , we define

$$(2) \quad s_{i,k} = kP + r_i$$

for  $i = 1, \dots, h$ . The inequality  $r_1 < r_2 < \dots < r_h$  implies that, for all  $k \geq 1$ ,

$$s_{1,k} < s_{2,k} < \dots < s_{h,k}.$$

We define the integers

$$(3) \quad S_k = \prod_{i=1}^h s_{i,k}$$

and

$$(4) \quad a_{i,k} = \frac{S_k}{s_{i,k}} = \prod_{\substack{j=1 \\ j \neq i}}^h s_{j,k}$$

for  $i = 1, \dots, h$ . For all  $k \geq 1$  we have  $S_k > 2^h$  and

$$a_{1,k} > a_{2,k} > \dots > a_{h,k}.$$

**Lemma 1.** *Define  $k_0$  by (1). Let  $r_1, \dots, r_h$  be a strictly increasing sequence of pairwise relatively prime positive integers, and let  $s_{1,k}, \dots, s_{h,k}$ ,  $S_k$ , and  $a_{1,k}, \dots, a_{h,k}$  be the positive integers defined by (2), (3), and (4), respectively. For all  $k \geq 1$ ,*

- (i) *the integers  $s_{1,k}, \dots, s_{h,k}$  are pairwise relatively prime,*
- (ii) *the integers  $a_{1,k}, \dots, a_{h,k}$  are relatively prime,*
- (iii)  *$S_k < S_{k+1}$  and, if  $k \geq k_0$ , then  $S_{k+1} < 2S_k$ .*

*Proof.* Let  $1 \leq i < j \leq h$ . If  $(s_{i,k}, s_{j,k}) = d > 1$ , then  $d$  divides both  $s_{i,k}$  and  $s_{j,k}$ , and so  $d$  divides  $s_{j,k} - s_{i,k} = r_j - r_i$ . If  $p$  is a prime divisor of  $d$ , then  $p$  divides  $r_j - r_i$  and so  $p$  divides  $P$ . Therefore,  $p$  divides both  $r_i$  and  $r_j$ , which is absurd, since  $(r_i, r_j) = 1$ . We conclude that  $s_{1,k}, s_{2,k}, \dots, s_{h,k}$  are pairwise relatively prime. This proves (i).

If  $\gcd(a_{1,k}, a_{2,k}, \dots, a_{h,k}) > 1$ , then there is a prime  $p$  that divides  $a_{i,k}$  for all  $i = 1, \dots, h$ . Since  $p$  divides  $a_{1,k}$ , it follows from (4) that  $p$  divides  $s_{j,k}$  for some  $j \in \{2, 3, \dots, h\}$ . Similarly, since  $p$  divides  $a_{j,k}$ , it follows from (4) that  $p$  divides  $s_{\ell,k}$  for some  $\ell \neq j$ . This is impossible, since  $(s_{j,k}, s_{\ell,k}) = 1$ , and so  $(a_{1,k}, a_{2,k}, \dots, a_{h,k}) = 1$ . This proves (ii).

To prove (iii), we use the inequality

$$s_{i,k} = kP + r_i < (k+1)P + r_i = s_{i,k+1}$$

for  $i = 1, \dots, h$  to obtain  $S_k < S_{k+1}$ . Since  $(x+z)/(y+z) < x/y$  if  $x > y > 0$  and  $z > 0$ , it follows that for  $k \geq k_0$  we have

$$\frac{s_{i,k+1}}{s_{i,k}} = \frac{(k+1)P + r_i}{kP + r_i} < \frac{(k+1)P}{kP} = 1 + \frac{1}{k} \leq 1 + \frac{1}{k_0} < 2^{1/h}$$

and so

$$\frac{S_{k+1}}{S_k} = \prod_{i=1}^h \frac{s_{i,k+1}}{s_{i,k}} < \left(1 + \frac{1}{k_0}\right)^h < 2.$$

This completes the proof.  $\square$

**Lemma 2.** *Let  $\{S_k\}_{k=0}^\infty$  be the strictly increasing sequence of positive integers defined by (3). Choose a positive integer  $k_1 \geq k_0$ . For every positive integer  $t$ , there is a unique integer  $\ell_t$  such that*

$$(5) \quad S_{k_1+\ell_t} \leq S_{k_1}^t < S_{k_1+\ell_{t-1}+1}.$$

*Then  $\ell_1 = 0$  and the sequence  $\{\ell_t\}_{t=1}^\infty$  is strictly increasing.*

*Proof.* We have  $S_{k_1} \leq S_{k_1}^1 < S_{k_1+1}$  and so  $\ell_1 = 0$ . By Lemma 1 (iii), we have  $S_{k+1} < 2S_k$  for all  $k \geq k_0$ . Since  $S_k > 2^h$  for all  $k \geq 1$ , it follows that

$$S_{k_1}^t < S_{k_1+\ell_t+1} < 2S_{k_1+\ell_t} < 2^h S_{k_1}^t < S_{k_1}^{t+1} < S_{k_1+\ell_{t+1}+1}$$

and so  $\ell_t < \ell_{t+1}$ . This completes the proof.  $\square$

**Lemma 3.** *Let  $r_1, \dots, r_h$  be a strictly increasing sequence of pairwise relatively prime positive integers, let  $k \geq 1$ , and let  $s_{1,k}, \dots, s_{h,k}$ ,  $S_k$ , and  $a_{1,k}, \dots, a_{h,k}$  be the integers defined by (2), (3), and (4), respectively. If*

$$(h-1)S_k < n \leq L$$

*then there exist nonnegative integers  $v_1, v_2, \dots, v_h$  such that*

$$a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h,k}v_h = n$$

*and*

$$v_i \leq \frac{L}{a_{i,k}}$$

*for  $i = 1, \dots, h$ .*

*Proof.* Since  $\gcd(a_{1,k}, a_{2,k}, \dots, a_{h,k}) = 1$ , there exist integers  $u_1, u_2, \dots, u_h$ , not necessarily nonnegative, such that

$$a_{1,k}u_1 + a_{2,k}u_2 + \dots + a_{h,k}u_h = n.$$

For each  $i = 1, 2, \dots, h-1$  there is a unique integer  $v_i \in \{0, 1, 2, \dots, s_{i,k} - 1\}$  such that

$$v_i \equiv u_i \pmod{s_{i,k}}.$$

Since  $S_k = a_{i,k}s_{i,k}$  for  $i = 1, \dots, h$ , we have

$$a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h-1,k}v_{h-1} \leq \sum_{i=1}^{h-1} a_{i,k}(s_{i,k} - 1) < (h-1)S_k$$

and

$$\begin{aligned} n &= a_{1,k}u_1 + a_{2,k}u_2 + \dots + a_{h-1,k}u_{h-1} + a_{h,k}u_h \\ &= a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h-1,k}v_{h-1} + a_{h,k}u_h + \sum_{i=1}^{h-1} a_{i,k}(u_i - v_i) \\ &= a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h-1,k}v_{h-1} + a_{h,k}u_h + S_k \sum_{i=1}^{h-1} \left( \frac{u_i - v_i}{s_{i,k}} \right) \\ &= a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h-1,k}v_{h-1} + a_{h,k}u_h + a_{h,k}s_{h,k} \sum_{i=1}^{h-1} \left( \frac{u_i - v_i}{s_{i,k}} \right) \\ &= a_{1,k}v_1 + a_{2,k}v_2 + \dots + a_{h-1,k}v_{h-1} + a_{h,k}v_h \end{aligned}$$

where

$$v_h = u_h + s_{h,k} \sum_{i=1}^{h-1} \left( \frac{u_i - v_i}{s_{i,k}} \right) \in \mathbf{Z}.$$

If  $n \geq (h-1)S_k$ , then

$$a_{h,k}v_h = n - \sum_{i=1}^{h-1} a_i v_i > n - (h-1)S_k \geq 0$$

and so  $v_h \geq 1$ . For  $i = 1, \dots, h$ , the inequality

$$0 \leq a_{i,k}v_i \leq n \leq L$$

implies that

$$v_i \leq \frac{L}{a_{i,k}}.$$

This completes the proof.  $\square$

**Theorem 1.** Let  $h \geq 2$  and  $k_1 \geq k_0$ . Let  $r_1, \dots, r_h$  be a strictly increasing sequence of pairwise relatively prime positive integers, and let  $s_{1,k}, \dots, s_{h,k}$ ,  $S_k$ , and  $a_{1,k}, \dots, a_{h,k}$  be the integers defined by (2), (3), and (4), respectively. Let  $\{\ell_t\}_{t=0}^\infty$  be the increasing sequence of nonnegative integers defined by (5). For  $t \geq 1$ , let

$$V(t) = \left\{ a_{i,k_1+\ell_t} v_i : i = 1, \dots, h \text{ and } 1 \leq v \leq \frac{(h-1)S_{k_1+\ell_{t+1}}}{a_{i,k_1+\ell_t}} \right\}.$$

The set

$$\mathcal{A} = [0, (h-1)S_{k_1}] \cup \bigcup_{t=1}^{\infty} V(t)$$

is a thin basis of order  $h$ .

*Proof.* Since  $0 \in \mathcal{A}$ , we have  $[0, (h-1)S_{k_1}] \subseteq \mathcal{A} \subseteq h\mathcal{A}$ . Recall that  $\ell_1 = 0$ . By Lemmas 1 and 2, the sequences  $\{S_k\}_{k=k_1}^\infty$  and  $\{\ell_t\}_{t=1}^\infty$  are strictly increasing. Therefore, for every  $n > (h-1)S_{k_1}$  there is a unique positive integer  $t$  such that

$$(h-1)S_{k_1+\ell_t} < n \leq (h-1)S_{k_1+\ell_{t+1}}.$$

By Lemma 3, the linear diophantine equation

$$a_{1,k_1+\ell_t}v_1 + a_{2,k_1+\ell_t}v_2 + \dots + a_{h,k_1+\ell_t}v_h = n$$

has a solution in integers  $v_1, \dots, v_h$  satisfying

$$0 \leq v_i \leq \frac{(h-1)S_{k_1+\ell_{t+1}}}{a_{i,k_1+\ell_t}}$$

for  $i = 1, 2, \dots, h$ . Since  $a_{i,k_1+\ell_t}v_i \in \{0\} \cup V(t) \subseteq \mathcal{A}$ , it follows that  $n \in h\mathcal{A}$  and so  $\mathcal{A}$  is a basis of order  $h$ .

Next, we prove that the basis  $\mathcal{A}$  is thin. For all  $t \geq 1$  there is a simple upper bound for the cardinality of the finite set  $V(t)$ :

$$|V(t)| \leq \sum_{i=1}^h \frac{(h-1)S_{k_1+\ell_{t+1}}}{a_{i,k_1+\ell_t}} < \frac{h(h-1)S_{k_1+\ell_{t+1}}}{a_{h,k_1+\ell_t}}.$$

Let  $A(x)$  be the counting function of the set  $\mathcal{A}$ . For  $x > (h-1)S_{k_1}$ , there is a unique positive integer  $t^*$  such that

$$(h-1)S_{k_1+\ell_{t^*}} < x \leq (h-1)S_{k_1+\ell_{t^*+1}}.$$

Using properties of the sequence  $\{\ell_t\}_{t=1}^\infty$ , we obtain

$$\begin{aligned}
A(x) &\leq A((h-1)S_{k_1+\ell_{t^*+1}}) \\
&\leq (h-1)S_{k_1} + \sum_{t=1}^{t^*} \frac{h(h-1)S_{k_1+\ell_{t+1}}}{a_{h,k_1+\ell_t}} + \sum_{t=t^*+1}^\infty \frac{h(h-1)S_{k_1+\ell_{t^*+1}}}{a_{h,k_1+\ell_t}} \\
&\leq (h-1)S_{k_1} + \sum_{t=1}^{t^*} \frac{h(h-1)S_{k_1}^{t+1}}{a_{h,k_1+\ell_t}} + \sum_{t=t^*+1}^\infty \frac{h(h-1)S_{k_1}^{t^*+1}}{a_{h,k_1+\ell_t}} \\
&< h(h-1)S_{k_1} \left( 1 + \sum_{t=1}^{t^*} \frac{S_{k_1}^t}{a_{h,k_1+\ell_t}} + \sum_{t=t^*+1}^\infty \frac{S_{k_1}^{t^*}}{a_{h,k_1+\ell_t}} \right) \\
&\leq h(h-1)S_{k_1} \left( 1 + \sum_{t=1}^{t^*} \frac{S_{k_1+\ell_t+1}}{a_{h,k_1+\ell_t}} + \sum_{t=t^*+1}^\infty \frac{S_{k_1+\ell_{t^*}+1}}{a_{h,k_1+\ell_t}} \right) \\
&\leq 2h(h-1)S_{k_1} \left( \sum_{t=0}^{t^*} \frac{S_{k_1+\ell_t}}{a_{h,k_1+\ell_t}} + \sum_{t=t^*+1}^\infty \frac{S_{k_1+\ell_{t^*}}}{a_{h,k_1+\ell_t}} \right).
\end{aligned}$$

We evaluate the sums separately. Let  $t \geq 1$ . Since  $P \geq r_h - r_1$  and  $k_1 + \ell_t \geq k_1 \geq k_0$ , we have

$$\frac{s_{h,k_1+\ell_t}}{s_{1,k_1+\ell_t}} = \frac{(k_1 + \ell_t)P + r_h}{(k_1 + \ell_t)P + r_1} < 1 + \frac{1}{k_1 + \ell_t} < 2^{1/h}$$

and so

$$s_{h,k_1+\ell_t} < 2^{1/h} s_{1,k_1+\ell_t}.$$

For all positive integers  $k$ , we define

$$\sigma_k = S_k^{1/h}.$$

Then  $\sigma_k > 2$  since  $S_k > 2^h$ .

The inequality  $(h-1)S_{k_1+\ell_{t^*}} < x$  implies that

$$\sigma_{k_1+\ell_{t^*}} \leq (h-1)^{1/h} \sigma_{k_1+\ell_{t^*}} < x^{1/h}.$$

We have

$$S_{k_1+\ell_t} = \sigma_{k_1+\ell_t}^h \leq S_{k_1}^t = \sigma_{k_1}^{ht} < S_{k_1+\ell_t+1} < 2S_{k_1+\ell_t} = 2\sigma_{k_1+\ell_t}^h$$

and so

$$\sigma_{k_1+\ell_t} \leq \sigma_{k_1}^t < 2^{1/h} \sigma_{k_1+\ell_t}.$$

Similarly, the inequality

$$s_{1,k_1+\ell_t}^h < S_{k_1+\ell_t} < s_{h,k_1+\ell_t}^h$$

implies that

$$s_{1,k_1+\ell_t} < \sigma_{k_1+\ell_t} < s_{h,k_1+\ell_t} < 2^{1/h} s_{1,k_1+\ell_t} < 2^{1/h} \sigma_{k_1+\ell_t}.$$

This gives an upper bound for the first sum:

$$\begin{aligned}
\sum_{t=0}^{t^*} \frac{S_{k_1+\ell_t}}{a_{h,k_1+\ell_t}} &= \sum_{t=0}^{t^*} s_{h,k_1+\ell_t} < 2^{1/h} \sum_{t=0}^{t^*} \sigma_{k_1+\ell_t} \\
&\leq 2^{1/h} \sum_{t=0}^{t^*} \sigma_{k_1}^t = 2^{1/h} \left( \frac{\sigma_{k_1}^{t^*+1} - 1}{\sigma_{k_1} - 1} \right) \\
&< 2 \cdot 2^{1/h} \sigma_{k_1}^{t^*} < 2 \cdot 4^{1/h} \sigma_{k_1+\ell_{t^*}} \\
&\leq 4x^{1/h}.
\end{aligned}$$

We used the fact that  $S_{k_1} > 2^h$  and so  $\sigma_{k_1} > 2$ .

To obtain an upper bound for the second sum, we observe that

$$\sigma_{k_1}^t < \sigma_{k_1+\ell_t+1} < 2^{1/h} \sigma_{k_1+\ell_t}$$

and so

$$\frac{1}{\sigma_{k_1+\ell_t}^{h-1}} < \frac{2^{(h-1)/h}}{\sigma_{k_1}^{t(h-1)}}.$$

Then

$$\begin{aligned}
\sum_{t=t^*+1}^{\infty} \frac{S_{k_1+\ell_{t^*}}}{a_{h,k_1+\ell_t}} &= S_{k_1+\ell_{t^*}} \sum_{t=t^*+1}^{\infty} \frac{s_{h,k_1+\ell_t}}{S_{k_1+\ell_t}} \\
&= 2^{1/h} S_{k_1+\ell_{t^*}} \sum_{t=t^*+1}^{\infty} \frac{\sigma_{k_1+\ell_t}}{\sigma_{k_1+\ell_t}^h} \\
&< 2^{1/h} S_{k_1+\ell_{t^*}} \sum_{t=t^*+1}^{\infty} \frac{1}{\sigma_{k_1+\ell_t}^{h-1}} \\
&< 2^{1/h} S_{k_1+\ell_{t^*}} \sum_{t=t^*+1}^{\infty} \frac{2^{(h-1)/h}}{\sigma_{k_1}^{t(h-1)}} \\
&= 2S_{k_1+\ell_{t^*}} \sum_{t=t^*+1}^{\infty} \frac{1}{\sigma_{k_1}^{t(h-1)}} \\
&= \frac{2S_{k_1+\ell_{t^*}}}{\sigma_{k_1}^{(t^*+1)(h-1)}} \left( \frac{\sigma_{k_1}^{h-1}}{\sigma_{k_1}^{h-1} - 1} \right) \\
&< \frac{4\sigma_{k_1}^{ht^*}}{\sigma_{k_1}^{(t^*+1)(h-1)}} \\
&< 4\sigma_{k_1}^{t^*} \\
&< 8x^{1/h}.
\end{aligned}$$

We conclude that

$$A(x) < 24h(h-1)S_{k_1}x^{1/h}$$

and so  $\mathcal{A}$  is a thin basis of order  $h$ . This completes the proof.  $\square$

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